The student materials for the first activity are complete: a Do Now page, a worksheet, and the supporting sketch. The follow-up activity, in which students will create the exponential functions corresponding to various tuning systems, is not yet finished, nor are these teacher notes finished for even the first activity.

EXPONENTIAL FUNCTIONS

There are many real-world situations that can be modeled using exponential and logarithmic functions, and in which mathematical modeling serves both to shed light on the situation and helps us to interact with it productively. Examples include pH scale used in chemistry, the decibel scale used to measure sound level and protect hearing, the Richter scale used to grade earthquakes and design safer structures, dealing with interest on investments and loans, and preventing or treating health risks related to bacterial growth or radioactive decay.

This pair of activities poses problems related to music, and specifically musical harmony, problems that are best addressed through the mathematics of exponential and logarithmic functions. They are designed to introduce exponential functions to students and to provoke students to create functions that help them understand and explain the relationships among different musical pitches.

The activities are sequenced, but each can be used independently, without the other.

The first activity is a hands-on exploration in which students pluck a stringed musical instrument to produce different pitches, evaluate what pitches sound harmonious, develop a conjecture about the mathematics of harmony, and use their conjecture to build a invent a musical scale. The only technology required is a stringed instrument. (A simple two-stringed instrument can easily be built from a piece of wood, several screws and nails, and a length of picture wire. Directions are below.) The activity is also supported by a sketch that has two strings whose relative length (and thus relative pitch) is easily modified. The objective of this lesson is for students to vary the ratio of the two pitches, to find patterns of consonance and dissonance, and to relate harmonious combinations to particular values of the pitch ratio expressed as a fraction. Students listen for harmony, for dissonance, and for the *beats* that result from very small differences between two pitches.

The second activity assumes that students already know that harmonious pitches have ratios that can be expressed as small integers such as 2/1, 3/2, and 4/3. Students use this knowledge to build sequences of notes and create functions to relate the notes to their pitches. They experiment with sequences based on different ratios of pitches from one note to the next, and evaluate the various possibilities in an effort to produce a set of notes, a musical scale, that includes as much harmony as possible. This quest takes two stages: in the first, they try using a particular integral ratio repeatedly—for instance, repeatedly using 2/1 allows you to play all octaves exactly but doesn't provide any intermediate notes, and using 3/2 allows you to play this important harmony but doesn't return to the octave.

Solving this problem requires compromise, with many different solutions proposed across the globe and over the years. All solutions about which I've read prioritize the octave interval. The solution based on Western classical music prioritizes the 3/2 ratio (the cycle of fifths), and modifies this solution to accommodate key changes, so that the same harmonies are available in every key. This solution has some very nice mathematics that is quite feasible for students to explore.

The student worksheet for the second activity, and most of the Teacher Notes section, remain to be written.

ACTIVITY OVERVIEW

To be written

OBJECTIVES

To be written

VOCABULARY

To be written (include harmony, frequency, octave – what else?)

PREPARATION

Home-made instruments like the one in the picture are not required for this activity, but they can be constructed very inexpensively. All that's required to build such an instrument is a board, several wood screws and nails, and some picture-hanging wire (5-lb strength works well). The screws should be slotted wood screws with a smooth section before the threads begin. The screws both anchor and adjust the piano wire, which is wrapped around the smooth section of the screw and then fixed in place by putting it through the slot and then wrapping some more. The screw must fit tightly into its hole in the wood so that it doesn't slip and allow the wire to go out of tune. The wire passes over the bent nails in the picture, fixing the ends of the vibrating portion of the wire. Use a screwdriver in the screw slots to adjust the pitch of the two wires to match each other. I've used a loose nail to form a movable fret used to adjust the length of one string; light thumb pressure on the wire to one side of this nail leaves the wire on the other side free to vibrate at the chosen length.

DO NOW

Answers and suggestions for the Do Now remain be written

INTRODUCE AND MODEL

The sketch Rational Harmony.gsp is useful for a whole-class introduction (and the summary at the end) as well as for students to use during their investigations, either in place of or in addition to the physical instrument.

The remainder of this section is to be written.

Re Q2 of Rational Harmony: If using homemade instruments, describe the technique for pressing without damping the sound.

Re Q4 of Rational Harmony: students may need to recheck the tuning in case it has shifted slightly.

EXPLORE

To be written

DISCUSS AND SUMMARIZE

ANSWERS

To be written

Q1

Rational Harmony Name:

DO NOW

1. Write these multiplication expressions using exponents: (Example: $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5$)

   a. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ b. $x \cdot x \cdot x \cdot x$ c. $(½) \cdot (½) \cdot (½)$

2. Write these expressions without exponents: (Example: $x^3 = x \cdot x \cdot x$)

   a. 5^4 b. a^6 c. $4 \cdot c^3$

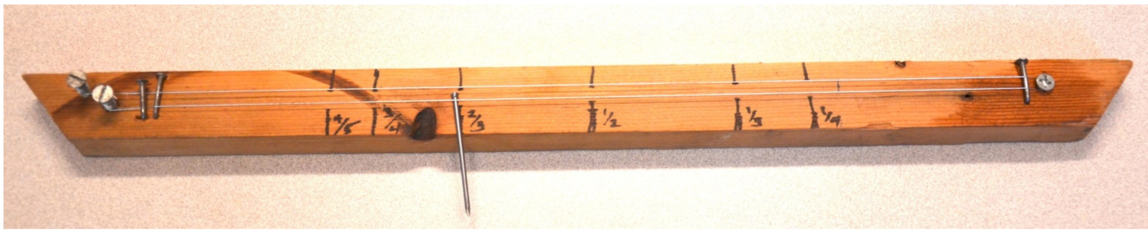
3. A full piano keyboard has 88 keys: 52 white keys and 36 black keys. The white keys are named using letters of the alphabet from A through G and then starting over again with A. This table lists the frequencies of all the A keys on a piano. For instance, when you press the A4 key, its string vibrates at 440 vibrations per second. Find the pattern, and fill in the missing frequencies.

Piano Key	Frequency
A0	
A1	
A2	
A3	220
A4	440
A5	
A6	1760
A7	

Challenge: Each A key on the piano is named for its octave number. For instance, the lowest A is called A0 because it is in octave 0. Can you use exponents to write a formula for each A key's frequency based on its octave number?

INVESTIGATE FUNCTION BEHAVIOR

First you will use a stringed musical instrument to listen to notes that are an octave apart. When notes are an octave apart, the string that makes the higher note vibrates twice as fast as the string that makes the lower note. You can use any stringed instrument: a guitar, a violin, a dulcimer, a lute, or even a homemade instrument like this one made of wood, nails, screws, and wire:



You can also use the sketch **Harmonic Ratios.gsp** to follow the same steps, before, after, or instead of the stringed instrument.

1. Adjust your instrument so that two of the strings are tuned to the same note. Pluck both strings at the same time, and listen to them. Then change the tuning of one string very very slightly and listen again.

Q1 When the two strings are very slightly out of tune, you should be able to hear something called "beats." Describe this sound in your own words.

2. Adjust your instrument so that two of the strings are tuned to the same note. Pluck both strings at the same time, and listen to them.

3. Change the tuning of one string very very slightly and listen again.

Q2 Shorten the vibrating part of the top string by pressing your finger on it. Does the pitch go up or down? In your own words, describe how the frequency changes as you shorten and lengthen the vibrating part of the top string.

4. Tune the two strings again to the same note Then measure the length of the string, and find the halfway point.

Q3 Record the length of the string and the length to the halfway point.

Q4 Press your finger at the halfway point of the top string, and play both strings. Try playing them one after the other, and try playing them at the same time. Describe the sounds you hear in your own words.

5. Make the top string a little longer than half the length of the bottom string, and play both strings. Then make it a little shorter and play both strings.

Q5 Describe the differences you hear when you adjust the top string to be less than half, exactly half, and more than half. If you listen very carefully, you should be able to hear the differences that result from very small changes in the length of the top string.

6. Use your measurement of the string length to find the place on the top string that will make the top string 2/3 the length of the bottom string.

Q6 Record the calculation you used to find the 2/3 point of the top string.

Q7 If a string half the length produces twice the frequency, what frequency would you expect from a string that's 2/3 the length? Explain your reasoning.

7. Press the top string at the 2/3 point and play both strings. Adjust where you press, and listen to the differences in the sound that result from the top string being a little longer than 2/3, exactly 2/3, and a little shorter than 2/3.

Q8 Describe the differences you hear when you adjust the top string to be less than 2/3, exactly 2/3, and more than 2/3. Listen carefully to detect the differences!

Q9 Make a conjecture about what notes sound harmonious together, and what sounds do not. Why do you think this conjecture makes sense?

Q10 Test your conjecture by trying some different notes and ratios than the ones you've already tried. When you find a note on the top string that sounds harmonious with the bottom string, measure the length of the top string and calculate the ratio of the two lengths. Describe your results.

Q11 Test your conjecture by tuning both strings to a different note. What do you notice? Do the same length ratios sound harmonious?

Q12 Of the different harmonious combinations you found, which sounds to you like the most harmonious? Describe what it was about this combination that sounded more harmonious than others.

Exponential Harmony Name:

DO NOW

A full piano keyboard has 88 keys: 52 white keys and 36 black keys. The white keys are named using letters of the alphabet from A through G and then starting over again with A. The Frequency column in the table below lists the frequencies of all the A keys on a piano. For instance, when you press the A4 key, its string vibrates at 440 vibrations per second.

1. The lowest A on the keyboard, A0, has a frequency of 27.5 vibrations/second. In the third column, fill in the missing calculations based on f_0, the frequency of A0 .

Piano Key	Frequency	Based on f_0	Using Exponents
A0	$f_0 = 27.5$	$f_0 \cdot 1$	
A1	$f_1 = 55$		
A2	$f_2 = 110$		$f_0 \cdot 2^2$
A3	$f_3 = 220$	$f_0 \cdot 2 \cdot 2 \cdot 2$	
A4	$f_4 = 440$		
A5	$f_5 = 880$		
A6	$f_6 = 1760$		
A7	$f_7 = 3520$		

2. In the fourth column, fill in the same missing calculations, using exponents to make them more readable.

3. The numbers 0 through 7 in the first column identify the octave number of the piano that contains the key, so the octave numbers go from 0 through 7. You can use n to stand for the octave number, and $f(n)$ to stand for the frequency for any A key on the keyboard. Using this notation, what calculation would you do to find $f(5)$?

$$f(5) =$$

4. Write a formula that you could use to find the frequency of any A key on the piano keyboard.

$$f(n) =$$

Exponential Harmony Name:

This worksheet, not yet written, is intended to guide students through the process of inventing and using several exponential functions. The first function, presaged in the Do Now above, is an exponential function that outputs the frequency for the A key in any octave. Students can create this function in Sketchpad, and then use its output in a second function that vibrates at the required frequency, and then make a Sound button to play the note.

The resulting Sound button is limited to playing octaves, with no notes in between, so the next step to making a Sketchpad piano is to use smaller steps than a full octave. A good start is to use the strongest rational harmony that students identified in the Rational Harmony activity (3/2). Such a function provides twice as many notes, but the notes are still pretty far apart. In addition, the resulting scale does not include the octave notes from before; in fact, it starts from A0, goes to E1, then to B1, to F#2, to C#3, and so forth.

To address the first problem, of steps that are too large, students can use smaller ratios on which to base the steps, perhaps 4/3, or 5/4. They can try several such ratios and see, for each ratio, how many intervals there are in an octave. But the second problem remains: the number of intervals in an octave is not a whole number. They can use more and more complicated ratios, but they will never hit the octave exactly.

The next step is to ask how they could divide the octave up exactly into two steps; investigating this problem, they will find that the required ratio is the square root of 2, and they will go on to determine that successive roots of 2 allow them to divide the octave into any desired number of intervals.

Finally, they can investigate a variety of numbers of intervals. This is best done in groups, so that different groups investigate different roots of 2. Each group should be responsible for reporting how close their notes come to the desired harmonious ratios. Students should decide which ratios are most important (e.g., 2/1, 3/2, 4/3, 5/4) and how to evaluate closeness (percentage error being one possible candidate).

With appropriate hints and questioning, students' explorations can result in their invention of a number of even-tempered scales, including the 12-tone scale of classical Western music. Students may also make good arguments for alternative scales with more or fewer tones, and they may propose modifications to the even temper to provide more accurate true fifths, fourths, and thirds.

The Piano page of the sketch allows them to play an even-tempered 12-interval scale and see the resulting ratios.

EXPLORE MORE

